

On Triple-Cut of Scattering Amplitudes

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It is analysed the triple-cut of one-loop amplitudes in dimensional regularisation within spinor-helicity representation. The triple-cut is defined as a difference of two double-cuts with the same particle contents, and a same propagator carrying, respectively, causal and anti-causal prescription in each of the two cuts. That turns out into an effective tool for extracting the coefficients of three-point functions (and higher-point ones) from one-loop amplitudes. The phase-space integration is oversimplified by using residues theorem to perform the integration on the spinor variables, via the holomorphic anomaly, and a trivial integration on the Feynman parameter. The results are valid for arbitrary values of dimensions.

1. Introduction

It is a well known fact that any one-loop amplitude with massless particles running in the loop, can be written, *via* standard Passarino-Veltman reduction, in terms of a basis of analytically known scalar integrals [1, 2, 3, 4], called master integrals (MI). Such a basis consists of box-, triangle-, and bubble-diagrams (I_4, I_3, I_2 respectively), which in four dimensions render the amplitude a combination of polylogarithms and rational terms.

To compute any amplitude, it is therefore sufficient to compute each of the rational coefficients entering that linear combination, and the principle of unitarity-based methods, as proposed by Bern, Dixon, Dunbar and Kosower [5], is the exploitation of the unitarity-cuts¹ of each MI, for reading its coefficient out of the amplitude.

Unitarity in four-dimension (4D) is sufficient to compute the polylogarithmic terms and the transcendental constants of one-loop amplitudes. By exploiting the analytic continuation of tree-amplitudes to complex spinors, initiated by Witten, Cachazo and Svrcek [6, 7], and the properties of the complex integration [8, 9, 10, 11], new techniques have generalised the cutting rules. On the one side, the *quadruple-cut* technique of Britto, Cachazo and Feng [12] yields the immediate computation of boxes' coefficient. On the other side, the polylogarithmic structure related to box-, triangle- and bubble-functions can be detected by a *double-cut* and computed by a novel way of performing the phase-space integral [13, 14], introduced by Britto, Buchbinder, Cachazo and Feng, in the context of Supersymmetry, that with Britto and Feng, we

further extended to deal with non-supersymmetric amplitudes, which combines the extraction of residues in spinor variables and the integration over a Feynman parameter.

However, on general grounds, amplitudes in nonsupersymmetric theories, like QCD, suffer of rational ambiguities that are not detected by the four-dimensional dispersive integrals. Therefore, in the very recent past several groups have developed new techniques focusing on the separate computation of the rational term of one-loop amplitudes. According to the combined *unitarity-bootstrap* approach, introduced by Bern, Dixon and Kosower in collaboration with Berger and Forde [15], the cut-containing terms computed by 4D-unitarity can provide corrective factors (due to factorisation constraints [16]) to a BCFW-like recurrence relation [17] for the reconstruction of the rational part, from the rational part of lower-point amplitudes. Xiao, Yang and Zhu have developed an optimized tool by tailoring the Passarino-Veltman reduction on the integrals that are responsible of the rational part of scattering amplitudes [18], giving rise to further refinements and new developments of algorithms for the tensor reduction of Feynman integrals like the *integrand decomposition* technique of Ossola, Papadopoulos and Pittau [19], and the *form-factors* method of Binoth, Guillet and Heinrich [20].

Alternatively, as it was realized by van Neerven [21], one can reconstruct the full amplitude from unitarity cuts in $D (= 4 - 2\epsilon)$ dimensions.

The unitarity-method introduced by Bern, Dixon, Dunbar, Kosower and Morgan [5, 22, 23] avoids the explicit evaluation of the phase-space integrals. It rather relies on the channel-by-channel reconstruction of the loop-integrand, by lifting the $\delta^{(+)}$ -functions to full propagators,

¹ In the following we equivalently use the terminology of *multiple cut* and *multi-particle cut*.

after having exploited as much as possible the simplification due to the on-shell cut-conditions. Then, by means of conventional techniques for loop-integrals, one obtains the reduction of the reconstructed-loop amplitude in terms of MI, enabling the extraction of their coefficients.

Brandhuber, McNamara, Spence and Travaglini [24] combined that technique of reconstructed-loop integrands with the generalised cutting rules, extending the effectiveness of the multiple-cuts, namely *quadruple*- and *triple*-cuts, from four to D dimensions. In particular the *quadruple*-cuts of reconstructed amplitudes yield the extraction of the coefficients of n -point MI, with $n \geq 4$; whereas the *triple*-cuts of reconstructed amplitudes yield the extraction of the coefficients of n -point MI, with $n \geq 3$, (which do have three denominators to be cut), and also of those 2-point MI that (do not have them, but) come from the tensor reduction of triangle-functions.

Recently, together with Anastasiou, Britto, Feng and Kunszt [25, 26], we have been able to extend, as well from the four dimensional case, the effectiveness of the true integration of the phase-space [13, 14], for the computation of the two-particle cut in D dimensions, by combining the extraction of residues in spinor variables and the parametric Feynman integration, convoluted with a further trivial parametric integration. Double-cuts in D dimensions can detect the coefficients of any n -point MI, with $n \geq 2$, which are what needed for the computation of any scattering amplitude (in absence of 0-point functions, the tadpoles).

In this letter, we present a new way for computing *triple*-cuts of dimensional regularised one-loop amplitudes. It enables the direct extraction of triangle- and higher-point-function coefficients from any one-loop amplitude in arbitrary dimensions. It combines the benefits of the double-cut integration of [13, 14, 25, 26] and of the exploitation of the on-shell cut-conditions [5, 22, 23, 24], through the idea of the inverse Cutkosky rule, already employed by Anastasiou and Melnikov [27, 28], to replace the third on-shell δ -function by the difference of two propagators,

$$(2\pi i)\delta(p^2 - \mu^2) \rightarrow \frac{1}{p^2 - \mu^2 + i0} - \frac{1}{p^2 - \mu^2 - i0}.$$

That yields an effective disentangling of the algebraic reduction of the integrand, achieved by trivial spinor algebra (Schouten identities), from the actual integrations which turn out to be oversimplified, when not trivialised.

Accordingly, the triple-cut is treated as a difference of two double-cuts with the same particle contents, and a same propagator carrying respectively causal and anti-causal prescription in each of the two cuts.

The triple-cut phase-space for massless particle in D dimensions is written as a convolution of a four-dimensional triple-cut of massive particle, and an integration over the corresponding mass parameter, which plays the role of a (-2ϵ) -dimensional scale [29].

As for the double-cut [25, 26], to perform the four-dimensional integration, we combine the method of spinor integration via the holomorphic anomaly of massive phase-space integrals, and an integration over the Feynman parameter. But, in the case of the triple-cut, after Feynman parametrisation, by combining back the two double-cuts, the parametric integration is reduced to the extraction of residues to the branch points in correspondence of the zeroes of a *standard quadratic function* (SQF'n) in the Feynman parameter. It is that SQF'n, or better, its roots that carry the analytic information characterizing each master-integral, therefore determining its own generalised cuts, hereafter called *master-cuts*, as it could be seen also from the seminal analysis by 't Hooft and Veltman of the generic scalar one-loop integrals [1].

The final integration over the dimensional scale parameter is mapped directly to triple-cut of master integrals, possibly with shifted dimensions [2, 3, 4].

The method hereby developed can be considered as one more dowel in the jigsaw of reconstructing any amplitude from its multiple generalised cuts along the lines of the Feynman Tree Theorem [30, 31] and the Veltman Largest Time Equation [32, 33]. In this spirit, one could now compute n -point ($n \geq 4$) coefficients from quadruple cuts, three-point coefficients from triple-cuts, and two-point coefficients from double-cuts, by avoiding the conventional tensor reduction. As it turns out, given the decomposition of any amplitude in terms of MI, the coefficient of any n -point MI can be recovered from the n -particle cut. Obviously, that n -particle cut may detect as well higher-point MI, which will appear with different analytic structures, for they come from the zeroes of a SQF'n specific of each diagram.

The triple-cut method hereby outlined can be applied to scattering amplitudes in gauge theories, and in Gravity as well. In particular, in the latter case, it is suitable for the analytic investigation of the so called “no-triangle hypothesis” for one-loop amplitudes in $\mathcal{N} = 8$ four-dimensional Supergravity, conjectured by Bern, Bjerrum-Bhor, and

Dunbar, and already confirmed together with Ita, Perkins and Risager at the 6- and 7-point level [34].

On the more speculative side, we think that the characterization of master integrals in terms of (the branch points corresponding to) the zeroes of the a standard function of the Feynman parameter might lead to a deeper understanding of the decomposition of one-loop amplitudes in terms of basic scalar integrals; and, together with the multi-particle cuts defined as iteration of (difference of) one-particle cuts, possibly, of their recursive behaviour.

Before proceeding with the analysis of the triple-cut, let us recap the double-cut technique introduced in [25, 26].

2. Double-Cut

We consider dimensional regularised one-loop amplitudes with massless propagators in the four-dimensional helicity (FDH) scheme, with external momenta living in four dimensions and the loop momentum living in a space with number of dimensions equal to $D (= 4 - 2\epsilon)$.

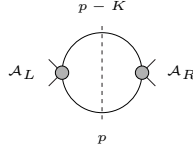


Figure 1. Double-Cut.

The discontinuity [27] of a generic one-loop amplitude in D -dimension is defined *via* the double-cut in Fig.1, corresponding to,

$$\mathcal{M} = \int d^{4-2\epsilon}\Phi \mathcal{A}_L^{\text{tree}} \times \mathcal{A}_R^{\text{tree}}, \quad (1)$$

$$d^{4-2\epsilon}\Phi = d^{4-2\epsilon}p \delta^{(+)}(p^2) \delta^{(+)}((K-p)^2) \quad (2)$$

where $d^{4-2\epsilon}\Phi$ is the Lorentz invariant two-body phase-space in D dimensions (D -LIPS), $\mathcal{A}_{L,R}^{\text{tree}}$ are tree-level amplitudes, and K is the total momentum across the cut.

Since the external momenta are in four dimensions, we can decompose the loop momentum as $p = L + \vec{\mu}$, where L is a 4-dimensional vector, and $\vec{\mu}$ is its (-2ϵ) -dimensional complement [29, 22]. As a consequence, the D -LIPS can be expressed as an integral over the dimensional scale μ^2 of a four-dimensional μ^2 -dependent discontinuity, equivalently written as

$$\int d^{4-2\epsilon}\Phi = \chi_K(\epsilon) \int_0^1 du u^{-1-\epsilon} \int d^4\Phi, \quad (3)$$

$$\chi_K(\epsilon) = \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)} \left(\frac{K^2}{4} \right)^{-\epsilon}, \quad (4)$$

$$u = \frac{4\mu^2}{K^2}, \quad (5)$$

$$\int d^4\Phi = \int d^4L \delta^{(+)}(L^2 - \mu^2) \delta^{(+)}((L-K)^2 - \mu^2). \quad (6)$$

In so doing, one can write the D -dimensional massless double-cut \mathcal{M} , essentially as an integral in u of a four dimensional (4D) massive double-cut, Δ ,

$$\mathcal{M} = \chi_K(\epsilon) \int_0^1 du u^{-1-\epsilon} \Delta, \quad (7)$$

with the 4D-discontinuity defined as,

$$\Delta = \int d^4\Phi \mathcal{A}_L^{\text{tree}}(p) \mathcal{A}_R^{\text{tree}}(p). \quad (8)$$

To perform the cut integration along the line of [10, 11, 12, 13, 14], the massive loop momentum L is decomposed into a linear combination of a light-like vector, ℓ_0 , and the time-like momentum-cut, K , [25, 26]

$$L = \ell_0 + zK, \quad \text{with } \ell_0^2 = 0. \quad (9)$$

After the rescaling [7],

$$\ell_0^{a\dot{a}} = t \ell^a \tilde{\ell}^{\dot{a}}, \quad (10)$$

the 4D-massive double-cut integration appears to be parametrised as,

$$\begin{aligned} \int d^4\Phi &= \int dz \frac{\delta(z-z_0)}{(1-2z)K^2} \int \langle \ell d\ell \rangle [\ell d\ell] \times \\ &\quad \int t^2 dt \delta\left(t - \frac{(1-2z)K^2}{\langle \ell | K | \ell \rangle}\right) \\ &= \int dz \delta(z-z_0) \int \frac{\langle \ell d\ell \rangle [\ell d\ell]}{\langle \ell | K | \ell \rangle} \times \\ &\quad \int t dt \delta\left(t - \frac{(1-2z)K^2}{\langle \ell | K | \ell \rangle}\right) \end{aligned} \quad (11)$$

$$z_0 = \frac{1 - \sqrt{1-u}}{2}, \quad (12)$$

where, z_0 is the proper root of the equation $z(1-z)K^2 - \mu^2 = 0$, as allowed by the $\delta^{(+)}$ -conditions. One can see the similarities between the massive and massless phase-space in four dimensions by comparing (11) and (182).

It is very important to notice that, due to the shift in (9), the spinor integration becomes light-like as required

by the method in [13, 14]. Accordingly, by means of basic spinor algebra, namely by Schouten identities, one can disentangle the dependence over $|\ell\rangle$ and $|\bar{\ell}\rangle$, and express the result of the t -integration as a combination of terms whose general form looks like,

$$\int t \, dt \, \delta\left(t - \frac{(1-2z)K^2}{\langle\ell|K|\ell\rangle}\right) \frac{\mathcal{A}_L^{\text{tree}}(\ell, z, t) \mathcal{A}_R^{\text{tree}}(\ell, z, t)}{\langle\ell|K|\ell\rangle} = \sum_i \mathcal{G}_i(|\ell\rangle, z) \frac{[\eta\ell]^n}{\langle\ell|P_1|\ell\rangle^{n+1}\langle\ell|P_2|\ell\rangle}, \quad (13)$$

where P_i can either be equal to K , or be a linear combination of external vectors, which depends on z , coming from (off-shell) propagators; and where \mathcal{G} 's depend solely on one spinor flavour, say $|\ell\rangle$ (and not on $|\bar{\ell}\rangle$), and may contain poles in $|\ell\rangle$ through factors like $1/\langle\ell|\Omega\rangle$ (with $|\Omega\rangle$ being a massless spinor, either associated to any of the external legs, say $|k_i\rangle$, or to the action of a vector on it, like $\not{P}|k_i\rangle$).

The explicit form of the vectors P_1 and P_2 in Eq.(13) is determining the nature of the 4D-double-cut, logarithmic or not, and correspondingly the topology of the diagram which is associated to. For easy of notation let us define the generic term in the *r.h.s.* of Eq.(13),

$$\mathcal{I}_i = \mathcal{G}_i(|\ell\rangle, z) \frac{[\eta\ell]^n}{\langle\ell|P_1|\ell\rangle^{n+1}\langle\ell|P_2|\ell\rangle} \quad (14)$$

Accordingly the 4D-discontinuity in Eq.(8) reads,

$$\Delta = \sum_i \int dz \, \delta(z - z_0) \int \langle\ell \, d\ell\rangle [\ell \, d\bar{\ell}] \mathcal{I}_i. \quad (15)$$

Let us distinguish among the two possibilities one encounters, in carrying on the spinor integration of \mathcal{I}_i :

1. $P_1 = P_2 = K$ (momentum across the cut). In this case, the result contains only the cut of a linear combination bubble-functions with external momentum K , and dimensions which might be or not shifted from the original value, D .
2. $P_1 = K$, $P_2 \neq K$, or $P_1 \neq P_2 \neq K$. In this case, the result can contain the cut of a linear combination of n -point functions with $n \geq 3$ and dimensions which might be, or not, shifted from the original value, D .

Since, in this letter, we are mainly interested in triangle-functions (and higher-point ones), or better, in their coefficients, we will focus on case 2.

2.1. Logarithmic Terms of 4D-Double-Cut

Let us assume that either $P_1 = K$, $P_2 \neq K$ or $P_1 \neq P_2 \neq K$. In such a situation, one proceeds by introducing a Feynman parameter, to write \mathcal{I}_i as,

$$\mathcal{I}_i = (n+1) \int_0^1 dx \, (1-x)^n \mathcal{G}_i(|\ell\rangle, z) \frac{[\eta\ell]^n}{\langle\ell|R|\ell\rangle^{n+2}}, \quad (16)$$

with

$$R = xP_1 + (1-x)P_2 \quad (17)$$

We can then proceed with the spinor integration of \mathcal{I}_i (the order of the integrations over the spinor variables and over the Feynman parameter can be exchanged).

First, one performs the integration over the $|\ell\rangle$ -variable by parts, using [13]

$$[\ell \, d\bar{\ell}] \frac{[\eta\ell]^n}{\langle\ell|P|\ell\rangle^{n+2}} = \frac{[d\ell \, \partial_{\bar{\ell}}]}{(n+1)} \frac{[\eta\ell]^{n+1}}{\langle\ell|P|\ell\rangle^{n+1}\langle\ell|P|\eta\rangle}. \quad (18)$$

Afterwards, the integration over the $|\ell\rangle$ -variable, by using Cauchy's residues theorem in the fashion of the holomorphic anomaly [9, 10, 11], yielding to,

$$\begin{aligned} \mathcal{F}_i &= \int \langle\ell \, d\bar{\ell}\rangle [\ell \, d\bar{\ell}] \mathcal{I}_i = \\ &= \int_0^1 dx \, (1-x)^n \int \langle\ell \, d\bar{\ell}\rangle [d\ell \, \partial_{\bar{\ell}}] \frac{\mathcal{G}_i(|\ell\rangle, z) [\eta\ell]^{n+1}}{\langle\ell|R|\ell\rangle^{n+1}\langle\ell|R|\eta\rangle} \\ &= \int_0^1 dx \, (1-x)^n \left\{ \frac{\mathcal{G}_i(R|\eta), z}{(R^2)^{n+1}} \right. \\ &\quad \left. + \sum_j \lim_{\ell \rightarrow \ell_{ij}} \langle\ell \, \ell_{ij}\rangle \frac{\mathcal{G}_i(|\ell\rangle, z) [\eta\ell]^{n+1}}{\langle\ell|R|\ell\rangle^{n+1}\langle\ell|R|\eta\rangle} \right\}, \quad (19) \end{aligned}$$

where $|\ell_{ij}\rangle$ are the simple poles of \mathcal{G}_i .

We may think to \mathcal{F}_i in (19) as decomposed into two pieces,

$$\mathcal{F}_i = \mathcal{F}_i^{(1)} + \mathcal{F}_i^{(2)} \quad (20)$$

with

$$\mathcal{F}_i^{(1)} \equiv \int_0^1 dx \, (1-x)^n \mathcal{G}_i(R|\eta), z) \frac{1}{(R^2)^{n+1}}, \quad (21)$$

$$\begin{aligned} \mathcal{F}_i^{(2)} &\equiv \int_0^1 dx \, (1-x)^n \\ &\quad \times \sum_j \lim_{\ell \rightarrow \ell_{ij}} \langle\ell \, \ell_{ij}\rangle \frac{\mathcal{G}_i(|\ell\rangle, z) [\eta\ell]^{n+1}}{\langle\ell|R|\ell\rangle^{n+1}\langle\ell|R|\eta\rangle}, \quad (22) \end{aligned}$$

The expressions (20, 21, 22) are the key point for the triple-cut construction, later discussed, therefore we will spend some words on it.

Let us observe that since \not{K} is linear in x , as from Eq.(17), R^2 is a quadratic function, the SQF'n, and can be written as,

$$R^2 = f(s_{ij}, z) (x - x_1) (x - x_2), \quad (23)$$

where f may depend on z and the invariants $s_{ij} = (k_i + k_j)^2$; and $x_{1,2}$ are the solutions of the equation $R^2 = 0$.

The key point is that R^2 is the signature of the master-cuts [1]. More properly, its roots x_1 and x_2 are irrational functions of the kinematic scales, s_{ij} and μ^2 , specific for each diagram, and allow to distinguish unequivocally among them. In fact, the cuts of any scalar master-integral are known from explicit calculation, and for triangle- and box-function one can see that the corresponding 4D-double-cut is proportional to the $\ln(x_1/x_2)$.

In particular the most general expression for R^2 , accounting for both $3m$ -triangle and $4m$ -box is a quadratic polynomial in x ,

$$R^2 = ax^2 + 2bx + c, \quad (24)$$

with coefficients

$$a = (P_1^2 - 2P_1 \cdot P_2 + P_2^2), \quad (25)$$

$$b = P_1 \cdot P_2 - P_2^2, \quad (26)$$

$$c = P_2^2, \quad (27)$$

and zeroes at the values

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - ac}}{a}. \quad (28)$$

Given three vectors, K_1, K_2, K_3 , bounded by momentum conservation, $K_1 + K_2 + K_3 = 0$, we can define the Källen λ -function

$$\lambda_{K_1, K_2, K_3} = (K_1^2)^2 + (K_2^2)^2 + (K_3^2)^2 - 2K_1^2 K_2^2 - 2K_1^2 K_3^2 - 2K_2^2 K_3^2. \quad (29)$$

We can, thus, write down the roots $x_{1,2}$ characterizing triangle- and box-function, by using the following expressions, according to the case.

• 3m-Triangle

For a generic 3-point function with external legs labeled with K_1, K_2, K_3 , and internal mass μ , one has:

$$\not{P}_2 = \not{K}_1; \quad (30)$$

$$\not{P}_1 = \frac{K_3^2 + 2zK_1 \cdot K_3}{K_1^2} \not{K}_1 + (1 - 2z) \not{K}_3; \quad (31)$$

$$P_2^2 = K_1^2; \quad (32)$$

$$P_1^2 = \frac{1}{K_1^2} \left(\frac{\mu^2}{K_1^2} \lambda_{K_1, K_2, K_3} + K_2^2 K_3^2 \right); \quad (33)$$

$$2P_1 \cdot P_2 = K_3^2 + K_2^2 - K_1^2. \quad (34)$$

• 4m-Box

For a generic 4-point function with external legs labeled with K_1, K_2, K_3, K_4 and internal mass μ , one has:

$$\not{P}_1 = \frac{K_1^2 - 2zK_{12} \cdot K_1}{K_{12}^2} \not{K}_{12} + (1 - 2z) \not{K}_1; \quad (35)$$

$$\not{P}_2 = \frac{K_4^2 + 2zK_{12} \cdot K_4}{K_{12}^2} \not{K}_{12} + (1 - 2z) \not{K}_4; \quad (36)$$

$$P_1^2 = \frac{1}{K_{12}^2} \left(\frac{\mu^2}{K_{12}^2} \lambda_{K_{12}, -K_2, -K_1} + K_{12}^2 K_1^2 \right); \quad (37)$$

$$P_2^2 = \frac{1}{K_{12}^2} \left(\frac{\mu^2}{K_{12}^2} \lambda_{K_{12}, K_3, K_4} + K_{12}^2 K_4^2 \right); \quad (38)$$

$$2P_1 \cdot P_2 = \frac{1}{K_{12}^2} \left[K_1^2 K_3^2 + K_2^2 K_4^2 + K_{12}^2 K_{41}^2 - 4\mu^2 (K_1^2 + K_4^2 - K_{41}^2) \right]. \quad (39)$$

Sub-cases for triangles and boxes with massless legs can be obtained by setting in the above expressions the corresponding momentum-square to zero.

The completion of the 4D-integration in (15), which reads,

$$\Delta = \sum_i \int dz \delta(z - z_0) \mathcal{F}_i, \quad (40)$$

can be achieved by merely substituting in the result of (20) the value $z = z_0$, given in (12) - hereafter is understood the equivalence of z and z_0 , and whenever z appears, z_0 should be intended.

Finally, in order to get the the discontinuity in D dimension (7), one should perform the very last integration over the dimensional parameter u ($= 4\mu^2/K^2$). Indeed, the u -integral is not to be carried out explicitly: it can either be expressed in terms of shifted dimension master-cut with coefficients not depending on ϵ [2, 3, 22, 4, 24]; or equivalently, as explained in [25, 26], it can be reduced *via* recurrence relations (obtained by integration-by-parts identities), to master cut in D dimensions and ϵ -dependent coefficients.

As a last remark, we observe that the 4D-massive discontinuity, which can be considered the kernel of the D -dimension integration (7), carries all the main information about the decomposition in terms of master-cuts [1]. Due to the role played by the integration over the dimensional variable u , the decomposition of the D -regularised cut-amplitude in terms of master-cuts in D dimensions, stems from the decomposition of the 4D-massive cut-amplitude, in terms of 4D-massive master-cut.

3. Triple-Cut

The triple-cut of a generic one-loop amplitude in D -dimension is defined as

$$\mathcal{N} = \int d^{4-2\epsilon}\Phi \delta^{(+)}((p+K_3)^2) \mathcal{A}_L \mathcal{A}_M \mathcal{A}_R, \quad (41)$$

where: $\int d^{4-2\epsilon}\Phi$ is the two-body phase space defined for the double-cut in (2); $(p+K_3)$ is the momentum corresponding to the extra cut-propagator; and $\mathcal{A}_{L,M,R}$ are the tree-level amplitudes forming the one-loop pattern.

By using the inverse of Cutkowsky rule [27, 28], to express the $\delta^{(+)}((p+K_3)^2)$ as a difference of two scalar propagators with opposite $i0$ -prescription, one can write the triple-cut

$$\begin{aligned} \mathcal{N} &= \frac{1}{(2\pi i)} \int d^{4-2\epsilon}\Phi \mathcal{A}_L \mathcal{A}_M \mathcal{A}_R \\ &\quad \times \left(\frac{1}{(p+K_3)^2 + i0} - \frac{1}{(p+K_3)^2 - i0} \right) \\ &= \frac{1}{(2\pi i)} (\mathcal{M}^+ - \mathcal{M}^-), \end{aligned} \quad (42)$$

as a difference of two double-cuts \mathcal{M}^\pm , with a same propagator carrying respectively a causal and anti-causal $i0$ -prescription in each of the two double-cuts, see Fig.2, where

$$\begin{aligned} \mathcal{M}^\pm &\equiv \int d^{4-2\epsilon}\Phi \mathcal{A}_L \times \frac{\mathcal{A}_M \mathcal{A}_R}{(p+K_3)^2 \pm i0} \\ &= \int d^{4-2\epsilon}\Phi \mathcal{A}_L \times \frac{\mathcal{A}_M \mathcal{A}_R}{(p+K_3)_\pm^2}. \end{aligned} \quad (43)$$

Hereafter, we keep track of the triple-cut propagator with the sub-index \pm . In this form, one can deal with \mathcal{M}^+ and \mathcal{M}^- as done in the previous section from Eq.(7) to Eq.(??), but by taking care of the presence of the $\pm i0$. Accordingly one has,

$$\mathcal{M}^\pm = \chi_K(\epsilon) \int_0^1 du u^{-1-\epsilon} \Delta^\pm \quad (44)$$

$$\begin{aligned} &= \chi_K(\epsilon) \int_0^1 du u^{-1-\epsilon} \sum_i \int dz \delta(z-z_0) \mathcal{F}_i^\pm \\ &= \chi_K(\epsilon) \int_0^1 du u^{-1-\epsilon} \int dz \delta(z-z_0) \\ &\quad \times \sum_i \left(\mathcal{F}_i^{(1,\pm)} + \mathcal{F}_i^{(2,\pm)} \right), \end{aligned} \quad (45)$$

with

$$\mathcal{F}_i^{(1,\pm)} \equiv \int_0^1 dx (1-x)^n \mathcal{G}_i(\not{R}[\eta], z) \frac{1}{(R_\pm^2)^{n+1}}, \quad (46)$$

$$\begin{aligned} \mathcal{F}_i^{(2,\pm)} &\equiv \int_0^1 dx (1-x)^n \\ &\quad \times \sum_j \lim_{\ell \rightarrow \ell_{ij}} \langle \ell \ell_{ij} \rangle \frac{\mathcal{G}_i(|\ell\rangle, z)}{\langle \ell | R | \ell \rangle_\pm^{n+1} \langle \ell | R | \eta \rangle_\pm} \cdot \end{aligned} \quad (47)$$

In this fashion, the D -dimensional massless triple-cut, \mathcal{N} , as well as for the double-cut, can be interpreted as a u -integral of a 4D-massive triple-cut, Θ ,

$$\mathcal{N} = \chi_K(\epsilon) \int_0^1 du u^{-1-\epsilon} \Theta, \quad (48)$$

where,

$$\begin{aligned} \Theta &= \frac{1}{(2\pi i)} (\Delta^+ - \Delta^-) \\ &= \frac{1}{(2\pi i)} \int dz \delta(z-z_0) \\ &\quad \times \sum_i \left(\mathcal{F}_i^{(1,+)} + \mathcal{F}_i^{(2,+)} - \mathcal{F}_i^{(1,-)} - \mathcal{F}_i^{(2,-)} \right). \end{aligned} \quad (49)$$

To reach the form for $\mathcal{F}_i^{(1,\pm)}$ and $\mathcal{F}_i^{(2,\pm)}$ given in (46) and (47) respectively, the Feynman parametrization should involve the extra cut-denominator $(p+K_3)^2$, which, after the shift (9) and the rescaling (10) has become,

$$(p+K_3)^2 \rightarrow \frac{t}{(1-2z)} \langle \ell | Q | \ell \rangle, \quad (50)$$

with

$$Q = (1-2z) \not{K}_3 + \frac{K_3^2 + 2zK \cdot K_3}{K^2} \not{K}. \quad (51)$$

In other words, the spinor algebra should be properly tailored to achieve a decomposition such that \not{R} appears to

$$\mathcal{A}_L(K) \begin{array}{c} p-K \\ \text{---} \\ \text{---} \\ \text{---} \\ p \end{array} \begin{array}{c} \mathcal{A}_M(K_2) \\ \text{---} \\ \text{---} \\ \text{---} \\ \mathcal{A}_R(K_3) \end{array} = \frac{1}{(2\pi i)} \left\{ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} +i0 \\ \text{---} \\ \text{---} \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} -i0 \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\}$$

Figure 2. Triple-cut in terms of two double-cuts, respectively with a causal propagator and an anti-causal propagator: \mathcal{A}_L , \mathcal{A}_M , and \mathcal{A}_R are tree level amplitudes, respectively depending on the external momenta K, K_2, K_3 .

be defined as a combination of two vectors, out of which one is \mathcal{Q} ,

$$\dot{R} = x\dot{P}_1 + (1-x)\mathcal{Q}. \quad (52)$$

In fact, R -type terms that after the Feynman parametrization do not contain \mathcal{Q} , therefore without any memory of the $i0$ -prescription, will just vanish from the triple-cut, once the two double-cut-like contributions, \mathcal{M}^\pm , will be combined back.

Before performing the x -integration, as one would do for the computation of a double-cut, in this case one can look at the full form of the triple-cut (42) in terms of $\mathcal{F}_i^{(1,\pm)}$ and $\mathcal{F}_i^{(2,\pm)}$,

$$\begin{aligned} \mathcal{N} &= \chi_K(\epsilon) \int_0^1 du u^{-1-\epsilon} \int dz \delta(z-z_0) \Theta \\ &= \chi_K(\epsilon) \int_0^1 du u^{-1-\epsilon} \int dz \delta(z-z_0) \\ &\quad \times \sum_i \left\{ \delta\mathcal{F}_i^{(1)} + \delta\mathcal{F}_i^{(2)} \right\}, \end{aligned} \quad (53)$$

with

$$\delta\mathcal{F}_i^{(1)} \equiv \frac{1}{(2\pi i)} \left(\mathcal{F}_i^{(1,+)} - \mathcal{F}_i^{(1,-)} \right) \quad (54)$$

$$\delta\mathcal{F}_i^{(2)} \equiv \frac{1}{(2\pi i)} \left(\mathcal{F}_i^{(2,+)} - \mathcal{F}_i^{(2,-)} \right). \quad (55)$$

The integration over the Feynman parameter, to be still performed both in $\delta\mathcal{F}_i^{(1)}$ and in $\delta\mathcal{F}_i^{(2)}$, is frozen by the presence of a δ -function, as follows,

$$\delta\mathcal{F}_i^{(1)} = \int_0^1 dx (1-x)^n \mathcal{G}_i(\dot{R}|\eta, z) \delta\left((R^2)^{n+1}\right), \quad (56)$$

$$\begin{aligned} \delta\mathcal{F}_i^{(2)} &= \sum_j \lim_{\ell \rightarrow \ell_{ij}} \langle \ell \ell_{ij} \rangle \mathcal{G}_i(|\ell|, z) [\eta \ell]^{n+1} \\ &\quad \times \int_0^1 dx (1-x)^n \delta\left(\langle \ell_{ij} | R | \ell_{ij} \rangle^{n+1} \langle \ell_{ij} | R | \eta \rangle\right). \end{aligned} \quad (57)$$

The expressions (53, 56, 57) represent the final form of a generic three-particle cut.

4. Examples

In the followings, we show some examples of the application of the triple-cut method. In Sec.4.1 and Sec.4.2, we compute the triple-cut of two master-integrals, namely the $1m$ -triangle, and the $0m$ -box; in Sec.4.3 and Sec.4.4 we compute the triple-cut of a linear triangle and a linear box integrals, respectively, to extract the coefficients of the $1m$ -triangle, and the $0m$ -box, in agreement with the results in the literature.

4.1. Scalar $1m$ -Triangle

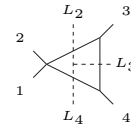


Figure 3. Triple-cut of a $1m$ -Triangle.

We consider the scalar integral represented in Fig.3, and associated to the triple-cut,

$$\begin{aligned} \mathcal{N}_{12|3|4} &= \int d^D \Phi \delta((L_2 - k_3)^2 - \mu^2) \\ &= \int d^D \Phi \delta(2L_2 \cdot k_3) \\ &\equiv \frac{1}{(2\pi i)} \int d^D \Phi \left\{ \frac{1}{(2L_2 \cdot k_3) + i0} \right. \\ &\quad \left. - \frac{1}{(2L_2 \cdot k_3) - i0} \right\} \end{aligned}$$

$$\begin{aligned}
&\equiv \frac{1}{(2\pi i)} \int d^D \Phi \left\{ \frac{1}{(2L_2 \cdot k_3)_+} - \frac{1}{(2L_2 \cdot k_3)_-} \right\} \\
&\equiv \frac{1}{(2\pi i)} \left\{ \mathcal{M}_{12}^{(+)} - \mathcal{M}_{12}^{(-)} \right\} \\
&= \chi_{12}(\epsilon) \int_0^1 du u^{-1-\epsilon} \frac{1}{(2\pi i)} \left\{ \Delta_{12}^{(+)} - \Delta_{12}^{(-)} \right\}
\end{aligned} \tag{58}$$

In so doing we have defined the triple-cut as a difference (modulo the overall factor $\frac{1}{(2\pi i)}$) of two double-cuts, each carrying memory of its own $i0$ -prescription.

The integration of the 4D-discontinuities, Δ^\pm , can proceed as well as for a double-cut. By defining,

$$\begin{aligned}
\mathcal{Q} &= (1-2z)\not{k}_3 - z\not{K}_{12} \\
&= (1-2z)\not{k}_3 + z\not{K}_{34} \\
&= (1-z)\not{k}_3 + z\not{k}_4,
\end{aligned} \tag{59}$$

after the shift (9) and the rescaling (10) with cut-momentum, $K_{12} (= -K_{34})$, one has

$$\begin{aligned}
\Delta_{12}^{(\pm)} &= -(1-2z) \int \langle \ell d\ell \rangle [\ell d\ell] \frac{1}{\langle \ell | K_{34} | \ell \rangle \langle \ell | Q | \ell \rangle_\pm} \\
&= -(1-2z) \int_0^1 dx \int \langle \ell d\ell \rangle [\ell d\ell] \frac{1}{\langle \ell | R | \ell \rangle_\pm^2}.
\end{aligned} \tag{60}$$

with

$$\not{R} = x\not{Q} + (1-x)\not{K}_{34} \tag{61}$$

Then one proceeds with

$$\Delta_{12}^{(\pm)} = -(1-2z) \int_0^1 dx \int \langle \ell d\ell \rangle [d\ell \partial_{\bar{\ell}}] \frac{[\eta \ell]}{\langle \ell | R | \ell \rangle_\pm \langle \ell | R | \eta \rangle} \tag{62}$$

which has a pole at $|\ell\rangle = \not{R}|\eta\rangle$. By taking its residue, one gets

$$\Delta_{12}^{(\pm)} = -(1-2z) \int_0^1 dx \frac{1}{R_\pm^2}, \tag{63}$$

with

$$R^2 = z(1-z) s_{12} (x-x_1)(x-x_2), \tag{64}$$

and

$$\begin{aligned}
x_1 &= \frac{1}{z} = \frac{2(1+\sqrt{1-u})}{u} \\
x_2 &= \frac{1}{1-z} = \frac{2(1-\sqrt{1-u})}{u}.
\end{aligned} \tag{65}$$

We remark that $x_{1,2}$ can be considered as the *characters* of the $1m$ -triangle.

We use at this stage the definition of the δ -function, yielding to the following expression for the 4D-massive triple-cut,

$$\begin{aligned}
\Theta_{12|3|4} &= \frac{1}{(2\pi i)} \left\{ \Delta_{12}^{(+)} - \Delta_{12}^{(-)} \right\} \\
&= -(1-2z) \int dx \delta(R^2) \\
&= -\frac{(1-2z)}{z(1-z)s_{12}} \int dx \delta((x-x_1)(x-x_2)) \\
&= -\frac{(1-2z)}{z(1-z)s_{12}} \left\{ \frac{1}{|x_1-x_2|} + \frac{1}{|x_2-x_1|} \right\} \\
&= -\frac{2(1-2z)}{z(1-z) s_{12} |x_1-x_2|} \\
&= -\frac{2}{s_{12}},
\end{aligned} \tag{66}$$

which is independent of u , therefore of μ^2 .

Finally, the complete D -dimensional triple-cut of a $1m$ -triangle $I_{3,1m}$ reads,

$$\begin{aligned}
\mathcal{N}_{12|3|4} &= \chi_{12}(\epsilon) \int_0^1 du u^{-1-\epsilon} \Theta_{12|3|4} \\
&= \chi_{12}(\epsilon) \int_0^1 du u^{-1-\epsilon} \frac{(-2)}{s_{12}}.
\end{aligned} \tag{67}$$

4.2. Scalar $0m$ -Box

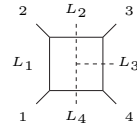


Figure 4. Triple-cut of a $0m$ -Box.

We consider the scalar integral represented in Fig.4, and associated to the triple-cut,

$$\begin{aligned}
N_{12|3|4} &= \int d^D \Phi \delta((L_2 - k_3)^2 - \mu^2) \frac{1}{(L_2 + k_2) - \mu^2} \\
&= \int d^D \Phi \delta(2L_2 \cdot k_3) \frac{1}{(2L_2 \cdot k_2)} \\
&= \chi_{12} \int_0^1 du u^{-1-\epsilon} \Theta_{12|3|4}
\end{aligned}$$

$$= \chi_{12} \int_0^1 du u^{-1-\epsilon} \frac{1}{(2\pi i)} \left\{ \Delta_{12}^{(+)} - \Delta_{12}^{(-)} \right\} \quad (68)$$

with

$$\Delta_{12}^{(\pm)} = \int d^4\Phi \frac{1}{(2L_2 \cdot k_3)_{\pm} (2L_2 \cdot k_2)} . \quad (69)$$

Since one has,

$$2L_2 \cdot k_3 = \frac{t}{(1-2z)} \langle \ell | Q_1 | \ell \rangle , \quad (70)$$

$$2L_2 \cdot k_2 = \frac{t}{(1-2z)} \langle \ell | Q_2 | \ell \rangle , \quad (71)$$

having defined

$$Q_1 \equiv (1-2z) \not{k}_3 + z \not{K}_{34} = (1-z) \not{k}_3 + z \not{k}_4 , \quad (72)$$

$$Q_2 \equiv (1-2z) \not{k}_2 + z \not{K}_{12} = (1-z) \not{k}_2 + z \not{k}_1 , \quad (73)$$

after the shift (9) and the rescaling (10) with cut-momentum, $K_{12} (= -K_{34})$, one can write,

$$\Delta_{12}^{(\pm)} = \int dz \frac{\delta(z-z_0)}{s_{12}} (1-2z) \int \frac{\langle \ell | d\ell \rangle [\ell | d\ell]}{\langle \ell | Q_1 | \ell \rangle_{\pm} \langle \ell | Q_2 | \ell \rangle} . \quad (74)$$

We give as understood the trivial z -integration, and perform the spinor integration as for a double-cut,

$$\begin{aligned} \Delta_{12}^{(\pm)} &= \frac{(1-2z)}{s_{12}} \int \langle \ell | d\ell \rangle [\ell | d\ell] \frac{1}{\langle \ell | Q_1 | \ell \rangle_{\pm} \langle \ell | Q_2 | \ell \rangle} \\ &= -\frac{(1-2z)}{s_{12}} \int_0^1 dx \int \langle \ell | d\ell \rangle [\ell | d\ell] \frac{1}{\langle \ell | R | \ell \rangle_{\pm}^2} , \\ &= -\frac{(1-2z)}{s_{12}} \int_0^1 dx \frac{1}{R_{\pm}^2} , \end{aligned} \quad (75)$$

with

$$\not{R} = x Q_2 + (1-x) Q_1 , \quad (76)$$

$$\begin{aligned} R^2 &= s_{23}(1-2z)^2 x^2 - s_{23}(1-2z)^2 x + s_{12}z(1-z) \\ &= s_{23}(1-2z)^2 (x-y_1)(x-y_2) , \end{aligned} \quad (77)$$

where

$$y_{1,2} = \frac{1}{2} \left(1 \pm \frac{\sqrt{Au+1}}{\sqrt{1-u}} \right) , \quad A = \frac{s_{13}}{s_{23}} . \quad (78)$$

We can therefore write the 4D-massive triple-cut as,

$$\Theta_{12|3|4} = \frac{1}{(2\pi i)} \left\{ \Delta_{12}^{(+)} - \Delta_{12}^{(-)} \right\}$$

$$\begin{aligned} &= -\frac{(1-2z)}{s_{12}} \int dx \delta(R^2) \\ &= -\frac{1}{s_{12}s_{23}(1-2z)} \int dx \delta((x-y_1)(x-y_2)) \\ &= -\frac{2}{s_{12}s_{23}(1-2z)|y_1-y_2|} \\ &= -\frac{2}{s_{12}s_{23}\sqrt{1+Au}} , \end{aligned} \quad (79)$$

which constitutes the integrand of the D -dimension triple-cut of a $0m$ -box function $I_{4,0m}$, finally reading as,

$$\begin{aligned} \mathcal{N}_{12|3|4} &= \chi_{12}(\epsilon) \int_0^1 du u^{-1-\epsilon} \Theta_{12|3|4} \\ &= \chi_{12}(\epsilon) \int_0^1 du u^{-1-\epsilon} \frac{(-2)}{s_{12}s_{23}\sqrt{1+Au}} . \end{aligned} \quad (80)$$

4.3. Linear Triangle

We consider the linear triangle integral appearing in Eqs.(3.45–3.47) of [24],

$$C^{\nu} = \int d^4 L_2 d^{-2\epsilon} \mu \frac{\mu^2 L_2^{\nu}}{D_1 D_2 D_3} \quad (81)$$

where

$$D_1 = L_2^2 - \mu^2 ; \quad (82)$$

$$D_2 = (L_2 + K_{12})^2 - \mu^2 ; \quad (83)$$

$$D_3 = (L_2 - k_3)^2 - \mu^2 . \quad (84)$$

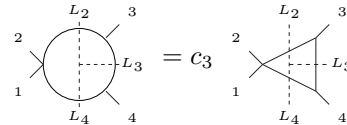


Figure 5. Triple-cut of a linear triangle in terms of the master triple-cut.

In particular let us consider the spinor sandwich $\langle 1|C|2 \rangle$ which appears in Eq.(3.43) of [24], and whose result, obtained by PV-reduction, reads

$$\langle 1|C|2 \rangle = \langle 1|3|2 \rangle \lambda , \quad (85)$$

with

$$\lambda = J_3 - \frac{2}{s_{12}} J_2 , \quad (86)$$

(overall factors understood, see [24] for details).

Let us reconstruct the coefficient of J_3 from the triple-cut integration, as depicted in Fig.5.

We begin with,

$$\begin{aligned}\mathcal{N}_{12|3|4} &= \int d^D \Phi \delta((L_2 - k_3)^2 - \mu^2) \mu^2 \langle 1|L_2|2 \rangle \\ &= \int d^D \Phi \delta(2L_2 \cdot k_3) \mu^2 \langle 1|L_2|2 \rangle \\ &= \chi_{12} \int_0^1 du u^{-1-\epsilon} \mu^2 \Theta_{12|3|4}\end{aligned}\quad (87)$$

where

$$\Theta_{12|3|4} \equiv \frac{1}{(2\pi i)} \left\{ \Delta_{12}^{(+)} - \Delta_{12}^{(-)} \right\} \quad (88)$$

with

$$\begin{aligned}\Delta_{12}^{(\pm)} &= \int d^4 \Phi \frac{\langle 1|L_2|2 \rangle}{(2L_2 \cdot k_3)_{\pm}} \\ &= \int dz \delta(z - z_0) s_{12}(1 - 2z)^2 \\ &\quad \times \int \langle \ell d \ell \rangle [\ell d \ell] \frac{\langle 1|\ell|2 \rangle}{\langle \ell|K_{12}|\ell \rangle^2 \langle \ell|Q_1|\ell \rangle_{\pm}}\end{aligned}\quad (89)$$

and

$$\begin{aligned}\mathcal{Q}_1 &= (1 - 2z)\not{k}_3 - z\not{K}_{12} \\ &= (1 - 2z)\not{k}_3 + z\not{K}_{34} \\ &= (1 - z)\not{k}_3 + z\not{k}_4,\end{aligned}\quad (90)$$

as in the case of the scalar triangle.

We give as understood the z -integration and proceed with the spinor integration as for a double-cut.

$$\begin{aligned}\Delta_{12}^{(\pm)} &= s_{12}(1 - 2z)^2 \int \langle \ell d \ell \rangle [\ell d \ell] \frac{\langle 1|\ell|2 \rangle}{\langle \ell|K_{12}|\ell \rangle^2 \langle \ell|Q_1|\ell \rangle_{\pm}} \\ &= s_{12}(1 - 2z)^2 \int_0^1 dx 2(1 - x) \\ &\quad \times \int \langle \ell d \ell \rangle [\ell d \ell] \frac{\langle 1|\ell|2 \rangle}{\langle \ell|R|\ell \rangle_{\pm}^3}\end{aligned}\quad (91)$$

where

$$\not{R} = x\mathcal{Q}_1 + (1 - x)\not{K}_{34}. \quad (92)$$

We proceed with,

$$\begin{aligned}\Delta_{12}^{(\pm)} &= s_{12}(1 - 2z)^2 \int_0^1 dx (1 - x) \\ &\quad \times \int \langle \ell d \ell \rangle [d \ell \partial_{\bar{\ell}}] \frac{\langle 1|\ell|2 \rangle^2}{\langle \ell|R|\ell \rangle_{\pm}^2 \langle \ell|R|2 \rangle}\end{aligned}\quad (93)$$

which has a pole at $|\ell \rangle = \not{R}|2 \rangle$, therefore

$$\Delta_{12}^{(\pm)} = s_{12}(1 - 2z)^2 \int_0^1 dx \frac{(1 - x) \langle 1|R|2 \rangle}{(R_{\pm}^2)^2}, \quad (94)$$

where

$$R^2 = z(1 - z) s_{12}(x - x_1)(x - x_2). \quad (95)$$

and $x_{1,2}$ given in (65). $\Delta_{12}^{(\pm)}$ can be written as,

$$\begin{aligned}\Delta_{12}^{(\pm)} &= s_{12} \langle 1|3|2 \rangle (1 - 2z)^3 \int_0^1 dx \frac{(1 - x)x}{(R_{\pm}^2)^2}, \\ &= \frac{\langle 1|3|2 \rangle (1 - 2z)^3}{s_{12} z^2 (1 - z)^2} \int_0^1 dx \frac{(1 - x)x}{(D_{\pm}^2)^2},\end{aligned}\quad (96)$$

where

$$D^2 = (x - x_1)(x - x_2). \quad (97)$$

We can use the following identity,

$$\frac{d}{dx} \frac{1}{D^2} = -\frac{2x - (x_1 + x_2)}{(D^2)^2} \quad (98)$$

to write

$$\begin{aligned}\Delta_{12}^{\pm} &= -\frac{\langle 1|3|2 \rangle (1 - 2z)^3}{s_{12} z^2 (1 - z)^2} \\ &\quad \times \int_0^1 dx \frac{(1 - x)x}{2x - (x_1 + x_2)} \frac{d}{dx} \frac{1}{D_{\pm}^2}.\end{aligned}\quad (99)$$

Therefore, we use the above result to obtain the 4D-massive triple-cut,

$$\begin{aligned}\Theta &\equiv \frac{1}{(2\pi i)} \left\{ \Delta_{12}^{(+)} - \Delta_{12}^{(-)} \right\} \\ &= -\frac{\langle 1|3|2 \rangle (1 - 2z)^3}{s_{12} z^2 (1 - z)^2} \int dx \frac{(1 - x)x}{2x - (x_1 + x_2)} \delta'(D^2) \\ &= -\frac{\langle 1|3|2 \rangle (1 - 2z)^3}{s_{12} z^2 (1 - z)^2} \int dx \frac{(1 - x)x}{2x - (x_1 + x_2)} \\ &\quad \times \left\{ \frac{\delta'(x - x_1)}{|x_1 - x_2|} + \frac{\delta'(x - x_2)}{|x_2 - x_1|} \right\} \\ &= \frac{\langle 1|3|2 \rangle (1 - 2z)^3}{s_{12} z^2 (1 - z)^2 |x_1 - x_2|} \\ &\quad \times \int dx \left\{ \delta(x - x_1) + \delta(x - x_2) \right\} \\ &\quad \times \frac{d}{dx} \frac{(1 - x)x}{2x - (x_1 + x_2)}\end{aligned}$$

$$\begin{aligned}
&= \frac{\langle 1|3|2 \rangle (1-2z)^3}{s_{12} z^2 (1-z)^2 |x_1 - x_2|} \\
&\quad \times \int dx \left\{ \delta(x - x_1) + \delta(x - x_2) \right\} \\
&\quad \times \frac{-2x^2 + 2(x_1 + x_2)x - (x_1 + x_2)}{(2x - (x_1 + x_2))^2} \\
&= \frac{(-2)}{s_{12}} \langle 1|3|2 \rangle. \tag{100}
\end{aligned}$$

Finally, one uses Eq.(87), to reconstruct the D -dimensional triple-cut,

$$\mathcal{N}_{12|3|4} = \langle 1|3|2 \rangle \chi_{12} \int_0^1 du u^{-1-\epsilon} \mu^2 \frac{(-2)}{s_{12}}, \tag{101}$$

out of which one can read the coefficient $c_3 = \langle 1|3|2 \rangle$, multiplying the cut of a $1m$ -triangle, see Eq.(67), in shifted dimensions, namely J_3 . The presence of terms like μ^{2m} is sterile for the 4D-integration, and it only affects the u -integral, see Eq.(5). As already said, the integration over u can be performed implicitly, by absorbing it in the redefinition of the integration measure for a value of dimensions which are shifted from the original one, $D \rightarrow D+2m$. That simply translates into the definition of the n -point J -type scalar integral, $J_n \equiv I_n[\mu^2]$, as having the same denominators as I_n and a single power of μ^2 up in the numerator [22, 24].

4.4. Linear Box

We consider the linear box integral,

$$B^\nu = \int d^4 L_2 d^{-2\epsilon} \mu \frac{\mu^2 L_2^\nu}{D_1 D_2 D_3 D_4} \tag{102}$$

where

$$D_1 = L_2^2 - \mu^2; \tag{103}$$

$$D_2 = (L_2 + K_{12})^2 - \mu^2; \tag{104}$$

$$D_3 = (L_2 - k_3)^2 - \mu^2; \tag{105}$$

$$D_4 = (L_2 + k_4)^2 - \mu^2. \tag{106}$$

In particular let us consider the spinor sandwich $\langle 1|B|2 \rangle$ which has the same value of $\langle 1|A|2 \rangle$ appearing in Eq.(3.43) of [24], and whose result, obtained by PV-reduction, reads

$$\langle 1|B|2 \rangle = \langle 1|3|2 \rangle \gamma, \tag{107}$$

with

$$\gamma = \frac{1}{2s_{13}} (s_{12} J_4 - 2J_3). \tag{108}$$

(overall factors understood, see [24] for details).

The above result, written in terms of $1m$ -triangle and $0m$ -box master integrals in shifted dimension, respectively J_3 and J_4 , can be entirely reconstructed from the triple-cut integration.

On general ground, any 4-point integral (therefore any amplitude) admits a decomposition in terms of master-integrals, as depicted in Fig.6. The coefficients $c_4, c_{3,1}$, and $c_{3,2}$ can be reconstructed from triple cuts. In particular, the integral $\langle 1|B|2 \rangle$ has two independent triple-cuts, namely $\mathcal{N}_{1|2|34}$ in Fig.7, and $\mathcal{N}_{12|3|4}$ in Fig.8, which we will discuss separately.

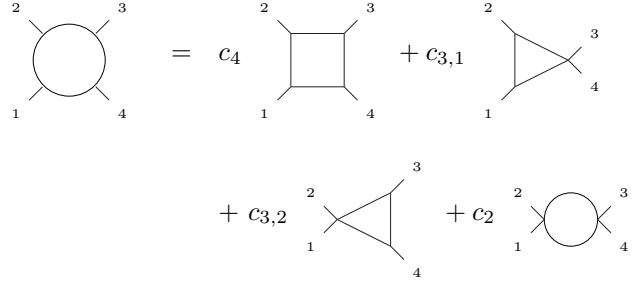


Figure 6. Decomposition of a 1-loop 4-point amplitude (or integral) in terms of a $0m$ -box, two $1m$ -triangles and a bubble, with rational coefficients c 's.

4.4.1. Triple-cut $\mathcal{N}_{1|2|34}$

The triple-cut integral corresponding to the *l.h.s.* of Fig.7, is defined as,

$$\begin{aligned}
\mathcal{N}_{1|2|34} &= \int d^D \Phi \delta((L_4 - k_1)^2 - \mu^2) \frac{\mu^2 \langle 1|L_4|2 \rangle}{(L_4 + k_4) - \mu^2} \\
&= \int d^D \Phi \delta(2L_4 \cdot k_1) \frac{\mu^2 \langle 1|L_4|2 \rangle}{(2L_4 \cdot k_4)} \\
&= \chi_{34} \int_0^1 du u^{-1-\epsilon} \mu^2 \Theta_{1|2|34} \tag{109}
\end{aligned}$$

where

$$\Theta_{1|2|34} = \frac{1}{(2\pi i)} \left\{ \Delta_{34}^{(+)} - \Delta_{34}^{(-)} \right\} \tag{110}$$

with

$$\Delta_{34}^{(\pm)} = \int d^4 \Phi \frac{\langle 1|L_4|2 \rangle}{(2L_4 \cdot k_1)_{\pm} (2L_4 \cdot k_4)}$$

$$\begin{aligned}
&= \int dz \frac{\delta(z - z_0)}{s_{34}(1 - 2z)} \int \langle \ell \, d\ell \rangle [\ell \, d\ell] \\
&\quad \times \int t^2 dt \, \delta\left(t - \frac{(1 - 2z)s_{34}}{\langle \ell | K_{34} | \ell \rangle}\right) \\
&\quad \times \frac{\langle 1 | L_4 | 2 \rangle}{(2L_4 \cdot k_1)_\pm (2L_4 \cdot k_4)} . \tag{111}
\end{aligned}$$

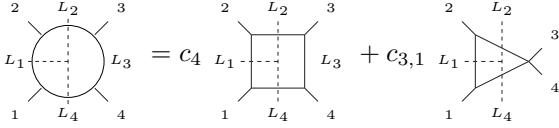


Figure 7. A triple-cut of a linear box in terms of the master triple-cuts.

Since,

$$2L_4 \cdot k_1 = \frac{t}{(1 - 2z)} \langle \ell | Q_1 | \ell \rangle , \tag{112}$$

$$2L_4 \cdot k_4 = \frac{t}{(1 - 2z)} \langle \ell | Q_2 | \ell \rangle , \tag{113}$$

with

$$Q_1 \equiv (1 - 2z) \not{k}_1 + z \not{K}_{12} = (1 - z) \not{k}_1 + z \not{k}_2 , \tag{114}$$

$$Q_2 \equiv (1 - 2z) \not{k}_4 + z \not{K}_{34} = (1 - z) \not{k}_4 + z \not{k}_3 , \tag{115}$$

one can write,

$$\begin{aligned}
\Delta_{34}^{(\pm)} &= \int dz \frac{\delta(z - z_0)}{s_{34}(1 - 2z)} \int \langle \ell \, d\ell \rangle [\ell \, d\ell] \\
&\quad \times \int t^2 dt \, \delta\left(t - \frac{(1 - 2z)s_{34}}{\langle \ell | K_{34} | \ell \rangle}\right) \\
&\quad \times \frac{t \langle 1 | \ell | 2 \rangle}{\frac{t}{(1 - 2z)} \langle \ell | Q_1 | \ell \rangle \pm \frac{t}{(1 - 2z)} \langle \ell | Q_2 | \ell \rangle} \\
&= \int dz \, \delta(z - z_0) (1 - 2z)^2 \mathcal{I}_{34} , \tag{116}
\end{aligned}$$

where

$$\mathcal{I}_{34} = \int \langle \ell \, d\ell \rangle [\ell \, d\ell] \frac{\langle 1 | \ell | 2 \rangle}{\langle \ell | K_{34} | \ell \rangle \langle \ell | Q_1 | \ell \rangle \pm \langle \ell | Q_2 | \ell \rangle} \tag{117}$$

By means of Schouten identities, and trivial spinor algebra, one can write,

$$\begin{aligned}
\mathcal{I}_{34} &= \int \langle \ell \, d\ell \rangle [\ell \, d\ell] \left(\frac{\langle 1 \ell \rangle [3 2]}{(1 - 2z) \langle 4 \ell \rangle \langle \ell | K_{34} | \ell \rangle \langle \ell | Q_1 | \ell \rangle [3 4]} \right. \\
&\quad - \frac{z \langle 1 \ell \rangle [3 2]}{(1 - 2z) \langle 4 \ell \rangle \langle \ell | Q_1 | \ell \rangle \langle \ell | Q_2 | \ell \rangle [3 4]} \\
&\quad + \frac{\langle 1 \ell \rangle [4 2]}{(1 - 2z) \langle 3 \ell \rangle \langle \ell | K_{34} | \ell \rangle \langle \ell | Q_1 | \ell \rangle [3 4]} \\
&\quad \left. - \frac{(1 - z) \langle 1 \ell \rangle [4 2]}{(1 - 2z) \langle 3 \ell \rangle \langle \ell | Q_1 | \ell \rangle \langle \ell | Q_2 | \ell \rangle [3 4]} \right) \\
&= \int_0^1 dx \int \langle \ell \, d\ell \rangle [\ell \, d\ell] \left(\frac{\langle 1 \ell \rangle [3 2]}{(1 - 2z) \langle 4 \ell \rangle \langle \ell | S_1 | \ell \rangle^2 [3 4]} \right. \\
&\quad + \frac{z \langle 1 \ell \rangle [3 2]}{(1 - 2z) \langle 4 \ell \rangle \langle \ell | R | \ell \rangle^2 [3 4]} \\
&\quad - \frac{\langle 1 \ell \rangle [4 2]}{(1 - 2z) \langle 3 \ell \rangle \langle \ell | S_1 | \ell \rangle^2 [3 4]} \\
&\quad \left. + \frac{(1 - z) \langle 1 \ell \rangle [4 2]}{(1 - 2z) \langle 3 \ell \rangle \langle \ell | R | \ell \rangle^2 [3 4]} \right) , \tag{118}
\end{aligned}$$

with

$$\not{S}_1 = -(1 - x) \not{K}_{34} + x \not{Q}_1 , \tag{119}$$

$$\not{R} = x \not{Q}_1 - (1 - x) \not{Q}_2 . \tag{120}$$

We can separate \mathcal{I}_{34} into two terms,

$$\mathcal{I}_{34} = \mathcal{I}_{34}^{(1)} + \mathcal{I}_{34}^{(2)} \tag{121}$$

where

$$\begin{aligned}
\mathcal{I}_{34}^{(1)} &= \int_0^1 dx \int \langle \ell \, d\ell \rangle [\ell \, d\ell] \left(\frac{\langle 1 \ell \rangle [3 2]}{(1 - 2z) \langle 4 \ell \rangle \langle \ell | S_1 | \ell \rangle^2 [3 4]} \right. \\
&\quad \left. - \frac{\langle 1 \ell \rangle [4 2]}{(1 - 2z) \langle 3 \ell \rangle \langle \ell | S_1 | \ell \rangle^2 [3 4]} \right) , \tag{122}
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{34}^{(2)} &= \int_0^1 dx \int \langle \ell \, d\ell \rangle [\ell \, d\ell] \left(\frac{z \langle 1 \ell \rangle [3 2]}{(1 - 2z) \langle 4 \ell \rangle \langle \ell | R | \ell \rangle^2 [3 4]} \right. \\
&\quad \left. + \frac{(1 - z) \langle 1 \ell \rangle [4 2]}{(1 - 2z) \langle 3 \ell \rangle \langle \ell | R | \ell \rangle^2 [3 4]} \right) , \tag{123}
\end{aligned}$$

each of which, being characterized by the presence of either S_1 or R , will lead unequivocally to triangle- and box-term respectively. Let's, therefore, discuss them separately.

• $\mathcal{I}_{34}^{(1)}$ term

To simplify the spinor integration, we use

$$\frac{[\ell \, d\ell]}{\langle \ell | S_1 | \ell \rangle^2} = [d\ell \, \partial_{\bar{\ell}}] \frac{[4 \, \ell]}{\langle \ell | S_1 | \ell \rangle \langle \ell | S_1 | 4 \rangle} \quad (124)$$

for the first term of $\mathcal{I}_{34}^{(1)}$, and

$$\frac{[\ell \, d\ell]}{\langle \ell | S_1 | \ell \rangle^2} = [d\ell \, \partial_{\bar{\ell}}] \frac{[3 \, \ell]}{\langle \ell | S_1 | \ell \rangle \langle \ell | S_1 | 3 \rangle} \quad (125)$$

for the second one, yielding

$$\begin{aligned} \mathcal{I}_{34}^{(1)} = \int_0^1 dx \frac{1}{(S_1^2)} & \left(- \frac{\langle 1 | S_1 | 4 \rangle [3 \, 2]}{(1-2z) \langle 4 | S_1 | 4 \rangle [3 \, 4]} \right. \\ & \left. - \frac{\langle 1 | S_1 | 3 \rangle [4 \, 2]}{(1-2z) \langle 3 | S_1 | 3 \rangle [3 \, 4]} \right) \end{aligned} \quad (126)$$

Since,

$$\langle 1 | S_1 | 4 \rangle = -(1-x+xz) \langle 1 | 3 | 4 \rangle, \quad (127)$$

$$\langle 4 | S_1 | 4 \rangle = s_{23}(1+A-Ax-xz+Axz), \quad (128)$$

$$\langle 1 | S_1 | 3 \rangle = -(1-x+xz) \langle 1 | 4 | 3 \rangle, \quad (129)$$

$$\langle 3 | S_1 | 3 \rangle = -s_{23}(-1-A+x-xz+Axz), \quad (130)$$

with

$$A = \frac{s_{24}}{s_{23}}, \quad (131)$$

one has,

$$\mathcal{I}_{34}^{(1)} = \frac{\langle 1 | 3 | 2 \rangle}{s_{23}(1-2z)} \int_0^1 dx \frac{f_1(x)}{(S_1^2)} \quad (132)$$

with

$$\begin{aligned} f_1(x) = & (1-x+xz) \\ & \times \left[\frac{1}{(1+A-Ax-xz+Axz)} \right. \\ & \left. - \frac{1}{(-1-A+x-xz+Axz)} \right], \end{aligned} \quad (133)$$

where we used, $\langle 1 | 4 | 2 \rangle = -\langle 1 | 3 | 2 \rangle$, due to momentum conservation. Since, see Eq.(65),

$$S_1^2 = s_{34} z(1-z) (x-x_1)(x-x_2), \quad (134)$$

and

$$x_1 = \frac{1}{z}; \quad x_2 = \frac{1}{1-z}, \quad (135)$$

the contribution to the 4D-massive triple-cut reads,

$$\begin{aligned} \delta \mathcal{I}_{34}^{(1)} &= \frac{\langle 1 | 3 | 2 \rangle}{s_{23} s_{34} z(1-z)(1-2z)} \\ &\times \int dx f_1(x) \delta((x-x_1)(x-x_2)) \\ &= -\frac{(1+A)}{A s_{23} s_{34}} \langle 1 | 3 | 2 \rangle \\ &= \frac{\langle 1 | 3 | 2 \rangle}{s_{23} s_{34}} \end{aligned} \quad (136)$$

where we used $s_{23} + s_{24} = -s_{34}$, due to momentum conservation.

• $\mathcal{I}_{34}^{(2)}$ term

The integral $\mathcal{I}_{34}^{(2)}$ has been defined in (123). We use

$$\frac{[\ell \, d\ell]}{\langle \ell | R | \ell \rangle^2} = [d\ell \, \partial_{\bar{\ell}}] \frac{[4 \, \ell]}{\langle \ell | R | \ell \rangle \langle \ell | R | 4 \rangle} \quad (137)$$

in the first term of $\mathcal{I}_{34}^{(2)}$, and

$$\frac{[\ell \, d\ell]}{\langle \ell | R | \ell \rangle^2} = [d\ell \, \partial_{\bar{\ell}}] \frac{[3 \, \ell]}{\langle \ell | R | \ell \rangle \langle \ell | R | 3 \rangle} \quad (138)$$

in the second one, yielding

$$\begin{aligned} \mathcal{I}_{34}^{(2)} = \int_0^1 dx \frac{1}{(R^2)} & \left[\frac{z \langle 1 | R | 4 \rangle [3 \, 2]}{(1-2z) \langle 4 | R | 4 \rangle [3 \, 4]} \right. \\ & \left. + \frac{(1-z) \langle 1 | R | 3 \rangle [4 \, 2]}{(1-2z) \langle 3 | R | 3 \rangle [3 \, 4]} \right]. \end{aligned} \quad (139)$$

Since,

$$\langle 1 | R | 4 \rangle = -z \langle 1 | 3 | 4 \rangle, \quad (140)$$

$$\langle 4 | R | 4 \rangle = -s_{23}(-x-z-Az+2xz), \quad (141)$$

$$\langle 1 | R | 3 \rangle = -(1-x-z+2xz) \langle 1 | 4 | 3 \rangle, \quad (142)$$

$$\langle 3 | R | 3 \rangle = -s_{23}(-1-A+x+z+Az-2xz), \quad (143)$$

one has,

$$\mathcal{I}_{34}^{(2)} = \frac{\langle 1 | 3 | 2 \rangle}{s_{23} (1-2z)} \int_0^1 dx \frac{f_2(x)}{(R^2)}, \quad (144)$$

with

$$\begin{aligned} f_2(x) = & \frac{z^2}{(-x-z-Az+2xz)} \\ & + \frac{(1-z)(1-x-z+2xz)}{(-1-A+x+z+Az-2xz)}, \end{aligned} \quad (145)$$

where we used, $\langle 1|4|2\rangle = -\langle 1|3|2\rangle$.

Since, see Eqs.(78),

$$R^2 = s_{23}(1-2z)^2 (x-y_1)(x-y_2), \quad (146)$$

where

$$y_{1,2} = \frac{1}{2} \left(1 \pm \frac{\sqrt{Au+1}}{\sqrt{1-u}} \right), \quad A = \frac{s_{13}}{s_{23}} = \frac{s_{24}}{s_{23}}, \quad (147)$$

the contribution to the 4D-massive triple-cut reads,

$$\begin{aligned} \delta \mathcal{I}_{34}^{(2)} &= \frac{\langle 1|3|2\rangle}{s_{23}^2 (1-2z)^3} \int dx f_2(x) \delta((x-y_1)(x-y_2)), \\ &= -\frac{\langle 1|3|2\rangle}{s_{23} s_{24} \sqrt{1+Au}}. \end{aligned} \quad (148)$$

By combining it with (136), one can now write down the result for the 4D-massive triple-cut,

$$\Theta_{1|2|34} = \left(\delta \mathcal{I}_{34}^{(1)} + \delta \mathcal{I}_{34}^{(2)} \right). \quad (149)$$

Finally, one uses Eq.(109) to reconstruct the D -dimensional triple-cut,

$$\begin{aligned} \mathcal{N}_{1|2|34} &= \chi_{12} \int_0^1 du u^{-1-\epsilon} \mu^2 \\ &\times \left\{ \frac{\langle 1|3|2\rangle}{s_{23} s_{34}} - \frac{\langle 1|3|2\rangle}{s_{23} s_{24} \sqrt{1+Au}} \right\} \end{aligned} \quad (150)$$

out of which one can read the coefficient

$$c_{3,1} = -\frac{\langle 1|3|2\rangle}{2 s_{23}}, \quad (151)$$

multiplying the cut of a $1m$ -triangle in shifted dimensions, namely J_3 , see Eq.(67); and the coefficient

$$c_4 = \frac{\langle 1|3|2\rangle s_{34}}{2 s_{24}}, \quad (152)$$

multiplying the cut of a $0m$ -box in shifted dimensions, namely J_4 , see Eq.(80).

4.4.2. Triple-cut $\mathcal{N}_{12|3|4}$

The second triple-cut needed for the reconstruction of the linear box-integral $\langle 1|B|2\rangle$ is depicted in to the *l.h.s.* of Fig.8, and is defined as,

$$\begin{aligned} \mathcal{N}_{12|3|4} &= \int d^D \Phi \delta((L_4 + k_4) - \mu^2) \frac{\mu^2 \langle 1|L_4|2\rangle}{(L_4 - k_1)^2 - \mu^2} \\ &= \chi_{12} \int_0^1 du u^{-1-\epsilon} \mu^2 \Theta_{12|3|4} \end{aligned} \quad (153)$$

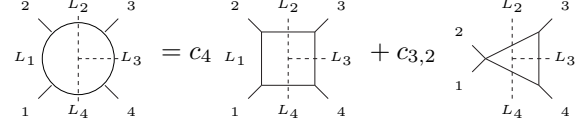


Figure 8. A triple-cut of a linear box in terms of the master triple-cuts.

where

$$\Theta_{12|3|4} = \frac{1}{(2\pi i)} \left\{ \Delta_{12}^{(+)} - \Delta_{12}^{(-)} \right\} \quad (154)$$

with

$$\Delta_{12}^{(\pm)} = \int dz \delta(z - z_0) (1-2z)^2 \mathcal{I}_{12}, \quad (155)$$

where

$$\mathcal{I}_{12} = \int \langle \ell d\ell \rangle [\ell d\ell] \frac{\langle 1|\ell|2\rangle}{\langle \ell|K_{34}|\ell\rangle \langle \ell|Q_1|\ell\rangle \langle \ell|Q_2|\ell\rangle_{\pm}} \quad (156)$$

By means of Schouten identities, different from the ones used for $\mathcal{N}_{1|2|34}$, one can write,

$$\begin{aligned} \mathcal{I}_{12} &= \int \langle \ell d\ell \rangle [\ell d\ell] \left(-\frac{\langle 1\ell \rangle}{(1-2z)\langle 2\ell \rangle \langle \ell|K_{34}|\ell\rangle \langle \ell|Q_2|\ell\rangle} \right. \\ &\quad \left. -\frac{(1-z)\langle 1\ell \rangle}{(1-2z)\langle 2\ell \rangle \langle \ell|Q_1|\ell\rangle \langle \ell|Q_2|\ell\rangle} \right) \\ &= \int_0^1 dx \int \langle \ell d\ell \rangle [\ell d\ell] \left(-\frac{\langle 1\ell \rangle}{(1-2z)\langle 2\ell \rangle \langle \ell|S_2|\ell\rangle^2} \right. \\ &\quad \left. +\frac{(1-z)\langle 1\ell \rangle}{(1-2z)\langle 2\ell \rangle \langle \ell|R|\ell\rangle^2} \right), \end{aligned} \quad (157)$$

where R has been defined in Eq.(120), and

$$\mathcal{S}_2 = (1-x)K_{34} + xQ_2, \quad (158)$$

We can separate \mathcal{I}_{12} into two terms,

$$\mathcal{I}_{12} = \mathcal{I}_{12}^{(1)} + \mathcal{I}_{12}^{(2)} \quad (159)$$

where

$$\mathcal{I}_{12}^{(1)} = -\int_0^1 dx \int \langle \ell d\ell \rangle [\ell d\ell] \frac{\langle 1\ell \rangle}{(1-2z)\langle 2\ell \rangle \langle \ell|S_2|\ell\rangle^2}, \quad (160)$$

$$\mathcal{I}_{12}^{(2)} = \int_0^1 dx \int \langle \ell d\ell \rangle [\ell d\ell] \frac{(1-z)\langle 1\ell \rangle}{(1-2z)\langle 2\ell \rangle \langle \ell|R|\ell\rangle^2}, \quad (161)$$

each of which, being characterized by the presence of either S_2 or R , will lead unequivocally to triangle- and box-term respectively. Let's, therefore, discuss them separately.

• $\mathcal{I}_{12}^{(1)}$ term

To simplify the spinor integration we use

$$\frac{[\ell \, d\ell]}{\langle \ell | S_2 | \ell \rangle^2} = [d\ell \, \partial_{\bar{\ell}}] \frac{[2 \, \ell]}{\langle \ell | S_2 | \ell \rangle \langle \ell | S_1 | 2 \rangle} \quad (162)$$

to write

$$\mathcal{I}_{12}^{(1)} = - \int_0^1 dx \frac{\langle 1 | S_2 | 2 \rangle}{(1-2z) \langle 2 | S_2 | 2 \rangle (S_2^2)} . \quad (163)$$

Since,

$$\langle 1 | S_2 | 2 \rangle = x(-1+2z) \langle 1 | 3 | 2 \rangle , \quad (164)$$

$$\langle 2 | S_2 | 2 \rangle = -s_{23}(-1-A+x-xz+Axz) , \quad (165)$$

one gets,

$$\mathcal{I}_{12}^{(1)} = - \frac{\langle 1 | 3 | 2 \rangle}{s_{23}} \int_0^1 dx \frac{f_3(x)}{(S_2^2)} , \quad (166)$$

where

$$f_3(x) = \frac{1}{(-1-A+x-xz+Axz)} . \quad (167)$$

Since,

$$S_2^2 = s_{34} z(1-z) (x-x_1)(x-x_2) , \quad (168)$$

and

$$x_1 = \frac{1}{z} ; \quad x_2 = \frac{1}{1-z} , \quad (169)$$

the contribution to the 4D-massive triple-cut reads,

$$\begin{aligned} \delta \mathcal{I}_{12}^{(1)} &= - \frac{\langle 1 | 3 | 2 \rangle}{s_{23} s_{34} z(1-z)} \\ &\quad \times \int dx f_3(x) \delta((x-x_1)(x-x_2)) , \\ &= \frac{(-1+A) \langle 1 | 3 | 2 \rangle}{A s_{23} s_{34}} \\ &= \frac{(s_{34}-s_{23}) \langle 1 | 3 | 2 \rangle}{s_{23} s_{24} s_{34}} . \end{aligned} \quad (170)$$

• $\mathcal{I}_{12}^{(2)}$ term

The integral $\mathcal{I}_{12}^{(2)}$ has been defined in (161). In this case, we use as well,

$$\frac{[\ell \, d\ell]}{\langle \ell | R | \ell \rangle^2} = [d\ell \, \partial_{\bar{\ell}}] \frac{[2 \, \ell]}{\langle \ell | R | \ell \rangle \langle \ell | R | 2 \rangle} \quad (171)$$

to have,

$$\mathcal{I}_{12}^{(2)} = \int_0^1 dx \frac{(1-z) \langle 1 | R | 2 \rangle}{(1-2z) \langle 2 | R | 2 \rangle (R^2)} . \quad (172)$$

Since,

$$\langle 1 | R | 2 \rangle = (1-x)(1-2z) \langle 1 | 3 | 2 \rangle , \quad (173)$$

$$\langle 2 | R | 2 \rangle = s_{23}(-A-x-z+Az+2xz) , \quad (174)$$

one gets,

$$\mathcal{I}_{12}^{(2)} = \frac{(1-z) \langle 1 | 3 | 2 \rangle}{s_{23}} \int_0^1 dx \frac{f_4(x)}{(R^2)} , \quad (175)$$

where

$$f_4(x) = \frac{(1-x)}{(-A-x-z+Az+2xz)} . \quad (176)$$

Therefore, given the expression of R^2 in (147), the contribution to the 4D-massive triple-cut reads,

$$\begin{aligned} \delta \mathcal{I}_{12}^{(2)} &= \frac{(1-z) \langle 1 | 3 | 2 \rangle}{s_{23}^2 (1-2z)^2} \int dx f_4(x) \delta((x-y_1)(x-y_2)) \\ &= - \frac{\langle 1 | 3 | 2 \rangle}{A s_{23}^2 \sqrt{1+Au}} \\ &= - \frac{\langle 1 | 3 | 2 \rangle}{s_{23} s_{24} \sqrt{1+Au}} . \end{aligned} \quad (177)$$

By combining it with (170), one can now write down the result for the 4D-massive triple-cut,

$$\Theta_{12|3|4} = \left(\delta \mathcal{I}_{12}^{(1)} + \delta \mathcal{I}_{12}^{(2)} \right) . \quad (178)$$

Finally, one uses Eq.(109) to reconstruct the D -dimensional triple-cut,

$$\begin{aligned} \mathcal{N}_{1|2|34} &= \chi_{12} \int_0^1 du u^{-1-\epsilon} \mu^2 \\ &\quad \times \left\{ \frac{\langle 1 | 3 | 2 \rangle (s_{34}-s_{23})}{s_{23} s_{24} s_{34}} - \frac{\langle 1 | 3 | 2 \rangle}{s_{23} s_{24} \sqrt{1+Au}} \right\} \end{aligned} \quad (179)$$

out of which one can read the coefficient

$$c_{3,2} = - \frac{\langle 1 | 3 | 2 \rangle (s_{34}-s_{23})}{2 s_{23} s_{24}} , \quad (180)$$

multiplying the cut of a $1m$ -triangle in shifted dimensions, namely J_3 , see Eq.(67); and the coefficient

$$c_4 = \frac{\langle 1 | 3 | 2 \rangle s_{34}}{2 s_{24}} , \quad (181)$$

multiplying the cut of a $0m$ -box in shifted dimensions, namely J_4 , see Eq.(80). We notice, that, as it should be, c_4 extracted from the triple-cut $\mathcal{N}_{12|3|4}$ is the same as obtained in (152) from $\mathcal{N}_{1|2|34}$.

The matching with the result of [24] re-written here in (107) can be confirmed. The coefficient of J_4 in (107) is exactly our c_4 , since $s_{34} = s_{12}$ and $s_{24} = s_{13}$. Whereas the coefficient of J_3 in (107) amounts to the sum $(c_{3,1} + c_{3,2})$, because, accidentally, the two 1m-triangles in Fig.6 can be expressed by the same function, J_3 .

5. Triple-Cut in Four Dimensions

Given the decomposition of a triple-cut in terms of two double-cuts, see Fig.2, in order to compute 4D-massless triple-cut, one has to use simply the two-particle massless phase-space, $d^4\phi$ [11, 13],

$$\int d^4\phi = \int \frac{\langle \ell d\ell \rangle [\ell d\ell]}{\langle \ell | K | \ell \rangle} \int t dt \delta\left(t - \frac{(K^2)}{\langle \ell | K | \ell \rangle}\right), \quad (182)$$

and perform the spinor integration along the line of [13, 14].

Triple-cut in four dimension allow the extraction of the coefficients of triangle- and box-functions from finite cuts of one-loop amplitudes, which enable the complete reconstruction of amplitudes, for example, in Supersymmetry and Gravity.

6. Conclusions

We have presented a new method for computing *triple cuts* of dimensional regularised one-loop amplitudes. It enables the direct extraction of triangle- and higher-point-function coefficients from any one-loop amplitude in arbitrary dimensions.

The triple-cut has been defined as a difference of two double-cuts, with the same particle contents and a same propagator carrying opposite $i0$ -prescription in each of the two cuts.

The three-particle D -dimensional phase-space measure is written as a standard convolution of a four-dimensional massive three-particle phase-space, and an integration over the corresponding mass parameter, which plays the role of the (-2ϵ) -dimensional scale.

The four-dimensional integration, in each double-cut, is carried on as, together with Anastasiou, Britto, Feng, and Kunszt, we have recently proposed, by combining the method of spinor integration via the holomorphic anomaly

of massive phase-space integrals, and an integration over the Feynman parameter. After Feynman parametrisation, by combining back the two double-cuts into the triple-cut, the parametric integration is reduced to the extraction of residues to the branch-points in correspondence of the zeroes of a standard function of the Feynman parameter, hereby called SQF'n, characterizing each master-integral. The final integration over the dimensional scale parameter is mapped directly to triple-cut of master integrals with shifted dimensions.

Along the line of the Feynman Tree Theorem for reconstructing any amplitude from its multiple generalised discontinuities, one can now compute n -point ($n \geq 4$) coefficients from quadruple cuts, three-point coefficients from triple-cuts, and two-point coefficients from double-cuts, by avoiding the conventional tensor reduction. Thus, given the decomposition of any amplitude in terms of MI, the coefficient of any n -point MI can be recovered from the all-channels n -particle cut. Each n -particle cut may detect as well higher-point MI, which will appear with different analytic structure, for it comes from specific zeroes of a standard quadratic function of the Feynman parameter.

The triple-cut method hereby outlined can be applied to scattering amplitudes in gauge theories, and in Gravity as well. In particular, in the latter case, it could be employed for the analytical investigation of the so called “no-triangle hypothesis” of $\mathcal{N} = 8$ Supergravity amplitudes, conjectured by Bern, Bjerrum-Bohr and Dunbar.

On the more speculative side, we think that the characterization of master integrals in terms of the zeroes of the corresponding SQF'n might lead to a deeper understanding of the decomposition of one-loop amplitudes in terms of basic scalar integrals, and, possibly, of their recursive behaviour.

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REFERENCES

1. G. 't Hooft, and M. Veltman, Nucl. Phys. B **153** (1979) 365.
2. Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299 [Erratum-ibid. B **318** (1993) 649] [arXiv:hep-ph/9212308].
3. Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B **412**, 751 (1994) [arXiv:hep-ph/9306240].
4. O. V. Tarasov, Phys. Rev. D **54**, 6479 (1996) [arXiv:hep-th/9606018].
5. Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425**, 217 (1994) [arXiv:hep-ph/9403226].
Z. Bern, et al., L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **435**:59 (1995) [hep-ph/9409265].
6. E. Witten, Commun. Math. Phys. **252**, 189 (2004) [arXiv:hep-th/0312171].
7. F. Cachazo, P. Svrcek and E. Witten, JHEP **0409** (2004) 006 [arXiv:hep-th/0403047].
8. F. Cachazo, P. Svrcek and E. Witten, JHEP **0410**, 074 (2004) [arXiv:hep-th/0406177].
9. F. Cachazo, P. Svrcek and E. Witten, JHEP **0410**, 077 (2004) [arXiv:hep-th/0409245].
10. F. Cachazo, arXiv:hep-th/0410077.
11. R. Britto, F. Cachazo and B. Feng, Phys. Rev. D **71**, 025012 (2005) [arXiv:hep-th/0410179].
12. R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B **725**, 275 (2005) [arXiv:hep-th/0412103].
13. R. Britto, E. Buchbinder, F. Cachazo and B. Feng, Phys. Rev. D **72**, 065012 (2005) [arXiv:hep-ph/0503132].
14. R. Britto, B. Feng and P. Mastrolia, Phys. Rev. D **73**, 105004 (2006) [arXiv:hep-ph/0602178].
15. Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D **71**, 105013 (2005) [arXiv:hep-th/0501240].
Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D **72**, 125003 (2005) [arXiv:hep-ph/0505055].
Z. Bern, L. J. Dixon and D. A. Kosower, hep-ph/0507005.
C. F. Berger, Z. Bern, L. J. Dixon, D. Forde and D. A. Kosower, hep-ph/0604195.
C. F. Berger, Z. Bern, L. J. Dixon, D. Forde and D. A. Kosower, hep-ph/0607014.
16. Z. Bern and G. Chalmers, Nucl. Phys. B **447**, 465 (1995) [arXiv:hep-ph/9503236].
17. R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B **715** (2005) 499 [arXiv:hep-th/0412308].
- R. Britto, F. Cachazo, B. Feng and E. Witten, Phys. Rev. Lett. **94** (2005) 181602 [arXiv:hep-th/0501052].
18. Z. G. Xiao, G. Yang and C. J. Zhu, hep-ph/0607015, hep-ph/0607016, hep-ph/0607017.
19. G. Ossola, C. G. Papadopoulos and R. Pittau, hep-ph/0609007.
20. T. Binoth, J. P. Guillet and G. Heinrich, hep-ph/0609054.
21. W. L. van Neerven, Nucl. Phys. B **268**, 453 (1986).
22. Z. Bern and A. G. Morgan, Nucl. Phys. B **467**, 479 (1996) [arXiv:hep-ph/9511336].
23. Z. Bern, et al., Phys. Lett. B **394**, 105 (1997).
24. A. Brandhuber, S. McNamara, B. J. Spence and G. Travaglini, JHEP **0510**, 011 (2005) [arXiv:hep-th/0506068].
25. C. Anastasiou, R. Britto, B. Feng, Z. Kunszt, and P. Mastrolia, arXiv:hep-ph/0609191.
26. C. Anastasiou, R. Britto, B. Feng, Z. Kunszt, and P. Mastrolia, in preparation.
27. R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960).
28. C. Anastasiou and K. Melnikov, Nucl. Phys. B **646**, 220 (2002) [arXiv:hep-ph/0207004].
29. G. Mahlon, [arXiv:hep-ph/9311213].
30. R. P. Feynman, Acta Phys. Polon. **24** (1963) 697.
R. P. Feynman, *Closed Loop and Tree Diagrams*, in J. R. Klauder, *Magic without magic*, San Francisco 1972, 355-375; in L.M. Brown (ed.): *Selected Papers of Richard Feynman*, 867-887.
R. P. Feynman, *Problems in Quantizing the Gravitational Field, and the Massless Yang-Mills Fields*, in J. R. Klauder, *Magic without magic*, San Francisco 1972, 377-408; in L.M. Brown (ed.): *Selected Papers of Richard Feynman*, 888-919.
31. A. Brandhuber, B. Spence, and G. Travaglini, JHEP **0601** (2006) 142 [arXiv:hep-th/0510253].
32. M. G. J. Veltman, Physica **29** (1963) 186.
33. E. Remiddi, Helv. Phys. Acta **54** (1982) 364.
34. Z. Bern, N. E. J. Bjerrum-Bohr, and D. C. Dunbar, JHEP **0505** (2005) 056 [arXiv:hep-th/0501137].
N. E. J. Bjerrum-Bohr, D. C. Dunbar, H. Ita, W. B. Perkins and K. Risager, arXiv:hep-th/0610043.